

AN APPROXIMATION THEOREM FOR MAPS BETWEEN TILING SPACES

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ABSTRACT. We show that every continuous map from one translationally finite tiling space to another can be approximated by a local map. If two local maps are homotopic, then the homotopy can be chosen so that every interpolating map is also local.

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1. INTRODUCTION

Many aspects of tiling theory, such as pattern-equivariant cohomology [K, KP], are built around local data. It might not matter what a tiling looks like near infinity, but it might matter crucially that a certain tile sits exactly *here*. Unfortunately, maps between tiling spaces may not preserve local data. Even topological conjugacies need not be local maps [Pet, RS]. To show that structures built from local data are actually topological invariants, we need to show that arbitrary maps between tiling spaces can be approximated by local maps, and that homotopies between local maps can be chosen to preserve locality at all times. These are the two main theorems of this paper. In addition, we show that all constructions can be chosen to preserve whatever discrete rotational symmetry exists.

For our purposes, a *tiling* is a decomposition of the plane (or, more generally, of \mathbb{R}^d) into a countable union of closed polygons (or polyhedra) that overlap only on their boundaries. These polygons are called *tiles*. One can consider more complicated shapes than polygons, but there is a standard trick, involving Voronoi cells [Pr], that converts non-polygonal tilings into polygonal tilings with the same mathematical properties.

A *patch* is a sub-collection of tiles in the tiling. For any tiling T and set $S \subset \mathbb{R}^d$, we let $[S]^T$ be the set of all tiles in T that intersect S . The *central patch of radius R* is the patch defined by $S = B_R(0)$, the closed ball of radius R around the origin. For any $x \in \mathbb{R}^d$, $T - x$ is a translate of T ; a neighborhood of the origin in $T - x$ looks like a neighborhood of the point x in T .

Two tilings are considered ϵ -close if they agree on $B_{1/\epsilon}(0)$, up to a rigid motion that moves points in $B_{1/\epsilon}(0)$ by ϵ or less. This rigid motion need not be a translation, but in most examples it is. A *tiling space* is a set of tilings that is complete in the tiling metric and is invariant under translations. The closure of the set of translates of any given tiling T is special kind of tiling space, called the *hull* of T .

A tiling space has *translational finite local complexity*, or is *translationally finite*, if the set of all patches of radius R is finite up to translation. A *tiling* is translationally finite if its hull is translationally finite. Translationally finite tiling spaces are necessarily compact.

If Ω_1 and Ω_2 are tiling spaces, we say a map $f : \Omega_1 \rightarrow \Omega_2$ is *local with radius R* if, whenever two tilings $T, T' \in \Omega_1$ have identical central patches of radius R , then $f(T)$ and $f(T')$ have identical central patches of radius 1. In other words, you don't need to know the behavior of T near infinity to specify the behavior of $f(T)$ near the origin.

For factor maps (i.e., maps that commute with translation), this is the analog of a sliding block code. On subshifts, continuous factor maps are always sliding block codes [LiM], but on tiling spaces, continuous factor maps need not be local [Pet, RS]. The problem has to do with small rigid motions. If T and T' agree on a large ball, then T and T' are close in the tiling metric, which means that $f(T)$ and $f(T')$ are close, which means that $f(T)$ and $f(T')$ agree on a large ball, *up to a small motion*.

In this paper, we show how to get rid of the small motion, although typically at the cost of not remaining a factor map. Indeed, we do not assume that our maps commute with translation to begin with! We merely show how to approximate arbitrary continuous maps with local continuous maps (Section 2), and how to approximate arbitrary homotopies between local maps with homotopies that preserve locality. These results are extensions, with streamlined proofs, of results first announced in [Ran].

Note that these results apply only to *translationally* finite tilings. If a tiling has finitely many patches of radius R up to Euclidean motion, but not up to translation (e.g., the pinwheel tiling [Rad]), then the averaging trick used to prove Theorem 1 breaks down, since we would have to average elements of a non-Abelian group. For a discussion of what can be proved for tilings with (rotational) finite local complexity, see [Ran].

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2. THE APPROXIMATION THEOREM

Theorem 1. *Let Ω_1 be the hull of a translationally finite and non-periodic tiling T_0 , let Ω_2 be a translationally finite tiling space, and let $f : \Omega_1 \rightarrow \Omega_2$ be a continuous map. For each $\epsilon > 0$ there exists a continuous local map $f_\epsilon : \Omega_1 \rightarrow \Omega_2$ such that f and f_ϵ differ only by a small translation. Specifically, there exists a continuous function $s_\epsilon : \Omega_1 \rightarrow \mathbb{R}^d$ such that, for each tiling T , $f_\epsilon(T) = f(T) - s_\epsilon(T)$ and $|s_\epsilon(T)| < \epsilon$.*

Proof. Ω_1 is the hull of T_0 and is translationally finite. This means that each patch P of a tiling $T \in \Omega_1$ is found somewhere in T_0 , say at position x , so P is the central patch of $T_0 - x$. Since f is continuous and Ω_1 is compact, f is uniformly continuous. Pick δ such that, if two tilings T_1, T_2 agree on $B_{1/\delta}(0)$, then $f(T_1)$ and $f(T_2)$ are within ϵ . Let $R = 2/\delta$. We will construct f_ϵ to be local with radius $R + \delta$.

T_0 is a single point in Ω_1 , but it is sometimes convenient to view T_0 as a marked copy of \mathbb{R}^d . For $x, y \in \mathbb{R}^d$, let $x \sim y$ if $[B_R(0)]^{T_0-x} = [B_R(0)]^{T_0-y}$. In other words, $x \sim y$ if the patch of radius R around x in T_0 looks like the patch around y . Let K_R be the quotient of \mathbb{R}^d by this equivalence relation. K_R is a branched d -manifold [BDHS, S] that parametrizes the possible patches of radius R . Since every patch of every tiling is found somewhere in T_0 , there is a natural projection $\pi : \Omega_1 \rightarrow K_R$ that sends each tiling to the description of its central patch.

K_R is a CW complex [BDHS], and is easily decomposed into disjoint cells of dimension up to d . For each cell C , pick a connected region $\tilde{C} \subset \mathbb{R}^d$ that represents C . That is, each point $p \in C$ is the equivalence class of a unique point $h(p) \in \tilde{C}$. Restricted to a single cell C , the map $h : K_R \rightarrow \mathbb{R}^d$ is continuous, but h may jump as we pass from one cell to another. Let $g = h \circ \pi : \Omega_1 \rightarrow \mathbb{R}^d$.

Let T be any tiling in Ω_1 . Since $\pi(T)$ describes the central patch of T , and since $g(T)$ is a point in T_0 whose patch of radius R agrees with the central patch of T , T and $T_0 - g(T)$ agree exactly on $B_R(0)$. This implies that $f(T)$ agrees with $f(T_0 - g(T))$ on $B_{1/\epsilon}(0)$, up to translation by up to ϵ . If ϵ is small, this translation is unique. Let $\tilde{s}_\epsilon(T)$ be the unique small element of \mathbb{R}^d such that $f(T) - \tilde{s}_\epsilon(T)$ agrees exactly with $f(T_0 - g(T))$ on $B_R(0)$. Let $\tilde{f}_\epsilon(T) = f(T) - \tilde{s}_\epsilon(T)$.

By construction, \tilde{f}_ϵ is local, but it may not be continuous, since h may not be continuous. We remedy this by convolving \tilde{f}_ϵ with a bump function. Let $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ be a smooth function of total integral 1, supported on $B_\delta(0)$. For each $y \in B_\delta$, $\tilde{f}_\epsilon(T - y)$ is a small translate of $f(T - y)$, and hence a small translate of $f(T)$. Let $\rho(T, y)$ be the unique small element of

\mathbb{R}^d such that $\tilde{f}_\epsilon(T - y) = f(T) - \rho(T, y)$. Let

$$(1) \quad s_\epsilon(T) = \int \phi(y) \rho(T, y) dy, \quad \text{and}$$

$$(2) \quad f_\epsilon(T) = f(T) - s_\epsilon(T).$$

It is clear that f_ϵ is continuous along a translational orbit. What remains is to show that f_ϵ is local. Suppose that T_1 and T_2 agree on $B_{R+\delta}$. $T_1 - y$ and $T_2 - y$ agree on $B_R(0)$, so $\tilde{f}_\epsilon(T_1 - y)$ and $\tilde{f}_\epsilon(T_2 - y)$ agree on a central patch. This means that $\rho(T_1, y) - \rho(T_2, y) = \alpha$, where α is the translation needed to take the central patch of $f(T_1)$ onto the central patch of $f(T_2)$. Integrating over y , we obtain $s_\epsilon(T_1) - s_\epsilon(T_2) = \alpha$, so the central patch of $f_\epsilon(T_1)$ agrees exactly with the central patch of $f_\epsilon(T_2)$. \square

3. THE HOMOTOPY THEOREM

Theorem 2. *Let $\Omega_{1,2}$ be as before. Let $f_0 : \Omega_1 \rightarrow \Omega_2$ and $f_1 : \Omega_1 \rightarrow \Omega_2$ be local maps. If $F : [0, 1] \times \Omega_1 \rightarrow \Omega$ is a homotopy between f_0 and f_1 , then there is another homotopy \tilde{F} between f_0 and f_1 such that each time slice is a local map from Ω_1 to Ω_2 .*

Proof. We apply the method outlined in the proof of Theorem 1 to each time slice f_t . Since the unit interval is compact, for any ϵ one can choose values of δ and R that work for every f_t . The resulting family of local maps $\tilde{F} : [0, 1] \times \Omega_1 \rightarrow \Omega$ gives a homotopy between $f_{0,\epsilon}$ and $f_{1,\epsilon}$. What remains is to construct a (local) homotopy between f_0 and $f_{0,\epsilon}$, and likewise between f_1 and $f_{1,\epsilon}$.

Since f_0 is already local, $\tilde{f}_{0,\epsilon} = f_0$, and $\rho(T, 0) = 0$ for every tiling T . For each $t > 0$, let $\phi_t(y) = t^{-d} \phi(y/t)$. When $t = 1$, we have our usual function ϕ , and equation (1) gives $f_{0,\epsilon}(T)$. As $t \rightarrow 0$, ϕ_t becomes a delta function, the integral approaches zero, and equation (1) gives a limiting value of $f_0(T)$. The same argument gives a local homotopy between f_1 and $f_{1,\epsilon}$. \square

4. ROTATIONS

Up to now we have been discussing tilings and the action of the translation group on them. For many tilings, such as the Penrose tiling, the rotation properties are also interesting. The following theorem extends Theorems 1 and 2 to that setting.

Theorem 3. *Let Ω_1 and Ω_2 be hulls of translationally finite tilings. Suppose that a finite subgroup G of $SO(d)$ acts naturally on Ω_1 and Ω_2 , and suppose that $f : \Omega_1 \rightarrow \Omega_2$ is a continuous maps that intertwines the actions of G . Then the approximation f_ϵ of Theorem 1, besides being local, can be chosen*

to intertwine the action of G . If f_0 and f_1 are homotopic maps $\Omega_1 \rightarrow \Omega_2$, and if each is local and each intertwines the action of G , then the homotopy between them can be chosen so that each f_t is local and intertwines the action of G .

Proof. Construct $s_\epsilon(T)$ exactly as in Theorem 1, only with a rotationally symmetric function $\phi(y)$, and then average over the group, defining $\bar{s}_\epsilon(T) = \frac{1}{|G|} \sum_{g \in G} g^{-1} s_\epsilon(gT)$ and $f_\epsilon(T) = f(T) - \bar{s}_\epsilon(T)$. That proves the first half of the theorem. Applying the same construction to the homotopy between f_0 and f_1 proves the second half of the theorem. \square

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